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Nonlinear Stability with Decay Rate for Traveling Wave Solutions of a Hyperbolic System with Relaxation*

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We derive decay estimates for small disturbances of smooth traveling wave solu-

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1. INTRODUCTION

We will discuss here the behavior of smooth traveling waves for a class of hyperbolic systems with relaxation of the type introduced by Lighthill and Whitham [4] and Richards [8] to model traffic flow on long highways. The systems we consider have the general form

$$u_t + f(u, v)_x = 0 \tag{1a}$$

$$v_t + g(u, v)_x = h(u, v). \tag{1b}$$

where f, g, h are known functions, with $h(u, v)$ typically given by

$$h(u, v) = \frac{v_*(u) - v}{\tau(u)},$$

$\tau(u)$ being a positive quantity representing the relaxation time. More generally, we will simply assume that, for all (u, v) concerned,

$$\frac{\partial h}{\partial v}(u, v) < 0 \tag{2}$$

with

$$h(u, v_*(u)) = 0 \tag{3}$$

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for some *equilibrium curve* $v = v_*(u)$ in the state space (u, v). We assume that f, g and h are smooth functions (three times continuously differentiable will be enough) and such that (1) is *strictly hyperbolic*, i.e., has real characteristics $\lambda_1(u, v) < \lambda_2(u, v)$ for all u, v concerned [3]. Hence, we must have

$$(f_u - g_v)^2 + 4 f_v g_u > 0$$

everywhere in the region of interest. It is also assumed that the equations in (1) are genuinely coupled, in the sense that

$$f_v(u, v) \neq 0 \tag{4}$$

for all (u, v) considered. Here, subscripts denote differentiation, so that

$$f_u \equiv \frac{\partial}{\partial u} f(u, v), \qquad f(u, v)_x \equiv \frac{\partial}{\partial x} f(u(x, t), v(x, t))$$

and so on. Equation (1a) is a conservation law for u [3], while the rate term $h(u, v)$ in (1b) tends to make v relax towards the equilibrium curve $v_*(u)$ (see (2), (3) above), acting as a source when v is less than $v_*(u)$ and as a sink otherwise. Thus, $h(u, v)$ introduces *relaxation effects* into the system (1). These effects add some amount of *diffusion*, as can be seen in the following example in traffic flow. Let $\rho(x, t)$ represent the density of cars at the location x and time t on a highway, i.e., the number of cars per unit length of the road, and let $w(x, t)$ denote the flow speed, that is, the speed of the car which reaches position x at time t. On a stretch of highway with no entries or exits, the number of cars is conserved, so that we have

$$\rho_t + (\rho w)_x = 0 \tag{5a}$$

which is our equation (1a) in this example. Now, what is the equation for $w(x, t)$? We take a model suggested by Lighthill, Whitham, and Richards [4, 8, 10]. Drivers on a highway continuously adjust their speed towards what they consider to be the ideal value under the local traffic conditions. We may possibly assume that this speed is the same for all drivers, say $w_*(x, t)$. Since they accelerate or decelerate their cars so that $w_*(x, t)$ is achieved, we may write

$$w_t + w w_x = -\frac{w - w_*}{\tau(\rho)} \tag{5b}$$

for some positive τ measuring the reaction time of the drivers, that is, the relaxation time. To get a complete model from (5a), (5b) above, we have to say how w_* is determined. First, we note that the ideal speed w_*

certainly depends on ρ, say $w_*(x, t) = W_*(\rho(x, t))$, with $W_*(\rho)$ decreasing as ρ increases from zero to its maximum value $\rho_{\max}$, when cars are bumper to bumper [10]. We thus assume that

$$W'_*(\rho) < 0, \; V''_*(\rho) < 0 \qquad \text{for} \quad 0 \leqslant \rho \leqslant \rho_{\max} \tag{5c}$$

where $V_*(\rho) \equiv \rho W_*(\rho)$. A second, more subtle effect can be included in w_*, which is precisely where diffusion is introduced in our system (5). A good driver checks the traffic conditions on the road ahead (and behind), compensating for possible changes in the flow. For example, if the number of cars is smaller ahead, he can afford a larger value for w_* than the number W_* dictated by the local density ρ alone. It looks then natural to take

$$w_* = W_*(\rho) - \kappa \frac{\rho_x}{\rho}, \tag{5d}$$

where κ is a positive quantity which tells how much the change in the number of cars is considered by the driver. A quick look at Eq. (5d) shows that κ has physical dimension $\mathrm{L}^2\mathrm{T}^{-1}$, like a diffusion coefficient. We can argue that κ gives indeed the amount of diffusion in the following way. According to (5b), the speed w tends to relax to w_*, so that, to a first approximation, we can view Eq. (5a) as

$$\rho_t + (\rho w_*(\rho))_x = 0,$$

or

$$\rho_t + V_*(\rho)_x = (\kappa \rho_x)_x,$$

a non-linear heat equation with diffusion coefficient given by κ. Thus, we should not be surprised to see small disturbances to constant, equilibrium states (ρ_0, w_0) of (5a)–(5d) traveling away with speed $\lambda_* = V'_*(\rho_0)$ and diffusing out as time increases, as pointed out by Liu [5]. Another consequence of diffusion is the existence of *smooth* shock waves, which will be the object of our study in this paper.

Many other systems exhibiting similar relaxation phenomena are known, see for example [5, 7, 9, 10] and references therein. A mathematical analysis of (1) has been done by Liu [5], where he describes the large-time pattern of its solutions. For the analysis of these systems, one often associates another system where certain quantities have already relaxed toward their equilibrium values, the so-called equilibrium system, in our case the scalar conservation law [5]

$$u_t + f_*(u)_x = 0, \qquad f_*(u) \equiv f(u, v_*(u)) \tag{6}$$

When (u, v) is close to equilibrium, the propagation of disturbances is governed by the characteristic speed of (6), i.e.,

$$\lambda_*(u) = f'_*(u) \tag{7}$$

see [5] for a detailed discussion. For strong non-equilibrium waves like shock waves, it is also shown in [5] that other speeds, different from the equilibrium characteristic λ_*, play an important role in the dynamics of the disturbances. Thus, in general, different propagation speeds are involved in the complex dynamics of the system described by equations (1) above.

Our interest here will be on *smooth traveling-wave solutions of* (1) *propagating at subcharacteristic speeds*, see (9) below. The existence of such waves is established in [5]; this is related to admissible shock discontinuities for the equation (6). If $(u, v)(x, t) = (\varphi, \psi)(x - \sigma t)$ is a traveling wave for (1) connecting constant states $(u_\pm, v_\pm) = (\varphi, \psi)(\pm\infty)$, propagating with (constant) speed σ, we must have

$$v_\pm = v_*(u_\pm),$$

since the only constant state solutions of (1) are equilibrium states. From (1a), we then get

$$\sigma \cdot (u_+ - u_-) = f_*(u_+) - f_*(u_-) \tag{8}$$

so that (u_-, u_+) satisfies the jump condition for the equilibrium equation (6); if this discontinuity is admissible (see [3, 6]) and its speed σ differs from $\lambda_1(u, v)$ and $\lambda_2(u, v)$ along $\sigma u - f(u, v) = \sigma u_\pm - f_*(u_\pm)$ for all u between u_- and u_+, then the existence of (φ, ψ) can be shown [5], and σ is subcharacteristic, i.e.,

$$\lambda_1(\varphi(\xi), \psi(\xi)) < \sigma < \lambda_2(\varphi(\xi), \psi(\xi)) \tag{9}$$

for all ξ (including $\xi = \pm\infty$), see [5] for details. Moreover, by combining a number of energy inequalities, Liu [5] was able to show that, for $|u_+ - u_-|$ small enough, these traveling waves are nonlinearly stable with respect to initial disturbances, provided they are sufficiently weak, in which case the only lasting effect on the original wave is a possible shifting according to the mass of the disturbances. For convenience of the reader, we review this result and give a sketch of the argument in the next section. Then, in Section 3, we refine this analysis following an approach similar to Kawashima–Matsumura [2] and derive new results which establish the decay rate of disturbances.

2. TRAVELING WAVE SOLUTIONS

In this section, we present a brief outline of the nonlinear stability analysis for traveling waves given by Liu [5]. The basic result discussed here is summarized in Theorem 2.1 below. Let $(\varphi, \psi)(x-\sigma t)$ be a smooth traveling wave of (1) with speed σ, and $(u, v)(x, t)$ be the solution of (1) corresponding to a slight perturbation of the initial profile for (φ, ψ), i.e.,

$$(u, v)(x, 0) = (\varphi, \psi)(x) + (\bar{u}, \bar{v})(x) \tag{10}$$

with $(\bar{u}, \bar{v})(\pm\infty) = 0$, $(\varphi, \psi)(\pm\infty) = (u_{\pm}, v_{\pm})$, $v_{\pm} = v_*(u_{\pm})$. More precisely, we assume that the initial disturbance $(\bar{u}, \bar{v})$ satisfies

$$\bar{u} \in H^2(\mathbf{R}) \cap L^1(\mathbf{R}), \qquad \bar{v} \in H^2(\mathbf{R}) \tag{11a}$$

and

$$\int_{-\infty}^{0} \left| \int_{-\infty}^{x} \bar{u}(y)\, dy \right|^2 dx + \int_{0}^{+\infty} \left| \int_{x}^{+\infty} \bar{u}(y)\, dy \right|^2 dx < \infty \tag{11b}$$

Because (1a) is a conservation law, we have

$$\int_{-\infty}^{+\infty} (u(x, t) - \varphi(x-\sigma t))\, dx = \int_{-\infty}^{+\infty} \bar{u}(x)\, dx,$$

so that we should not expect $(u, v)(x, t)$ to tend to $(\varphi, \psi)(x-\sigma t)$ as $t \to \infty$ unless the initial disturbance $\bar{u}(x)$ had zero integral on the line [1]. However, observing that, for fixed x_0,

$$\int_{-\infty}^{+\infty} (\varphi(x+x_0) - \varphi(x))\, dx = x_0 \cdot (u_+ - u_-),$$

we see that

$$\int_{-\infty}^{+\infty} (u(x, t) - \varphi(x+x_0-\sigma t))\, dx = \int_{-\infty}^{+\infty} \bar{u}(x)\, dx - x_0 \cdot (u_+ - u_-) = 0$$

provided we take x_0 to be

$$x_0 = \frac{1}{u_+ - u_-} \int_{-\infty}^{+\infty} \bar{u}(x) \tag{12}$$

That is, for such x_0, we can expect to show that

$$(u, v)(x, t) \to (\varphi, \psi)(x+x_0-\sigma t) \qquad \text{as} \quad t \to \infty \tag{13}$$

Thus, the original wave (φ, ψ) is shifted by the amount x_0 given by (12) due to the disturbance. This is in vivid contrast with the corresponding behavior of expansion waves, which interact very weakly with the disturbances [5]. Also, it turns out that, in the case of traveling waves, disturbances might decay quite fast as $t \to \infty$, provided they are sufficiently localized in space. In Section 3, we show that this is indeed the case; for now, we will simply outline the basic steps in proving (13), following [5]. It is convenient at this point to introduce the following notation: for given $a \in \mathbf{R}$, let $\mathcal{S}_a$ be the shift operator $(\mathcal{S}_a \phi)(x) = \phi(x+a)$, ϕ an arbitrary function defined on the real line. We are then to study the behavior of $(u, v)(\cdot, t) - (\mathcal{S}_{x_0 - \sigma t} \varphi, \mathcal{S}_{x_0 - \sigma t} \psi)$ as $t \to \infty$. From (11a), (11b), we can easily check that

$$(u, v)(\cdot, 0) - (\mathcal{S}_{x_0} \varphi, \mathcal{S}_{x_0} \psi) \in H^2(\mathbf{R}) \tag{14a}$$

and

$$\int_{-\infty}^{+\infty} \left| \int_{-\infty}^{x} (u(y, 0) - \mathcal{S}_{x_0} \varphi(y))\, dy \right|^2 dx < \infty \tag{14b}$$

When these quantities (14a), (14b) are small, we can derive energy estimates related to (13), leading to the following result [5]:

THEOREM 2.1 (Liu, 1987). *Consider a traveling wave solution* $(\varphi, \psi)(x - \sigma t)$ *for the hyperbolic system* (1), *with* $(\varphi, \psi)(\pm\infty) = (u_\pm, v_\pm)$, $v_\pm = v_*(u_\pm)$, *propagating at a subcharacteristic speed* σ, *i.e.,* $\lambda_1((\varphi, \psi)(\xi)) < \sigma < \lambda_2((\varphi, \psi)(\xi))$ *for all* $-\infty \leqslant \xi \leqslant +\infty$, *where* λ_1 *and* λ_2 *denote the characteristic values for* (1).

Then there exist positive constants δ, ε *such that the following is true whenever* $|u_+ - u_-| \leqslant \delta$:

Let $(\bar{u}, \bar{v})$ *satisfy* (11a) *and* (11b), *and let* $(u, v)(x, t)$ *be the solution of* (1) *corresponding to the initial data* (10), *i.e.,*

$$u(x, 0) = \varphi(x) + \bar{u}(x)$$

$$v(x, 0) = \psi(x) + \bar{v}(x).$$

Let

$$z(x, t) = \int_{-\infty}^{x} (u(y, t) - \mathcal{S}_{x_0} \varphi(y - \sigma t))\, dy$$

$$w(x, t) = v(x, t) - \mathcal{S}_{x_0} \psi(x - \sigma t),$$

where x_0 *is given by* (12).

Then (u, v) *is defined for all* $t \geqslant 0$ *and*

$$\|u(\cdot, t) - \mathscr{S}_{x_0 - \sigma t}\varphi\|_{H^2} + \|v(\cdot, t) - \mathscr{S}_{x_0 - \sigma t}\psi\|_{H^2} = O(1)(\|z(\cdot, 0)\|_{H^3} + \|w(\cdot, 0)\|_{H^2})$$

and

$$\|u(\cdot, t) - \mathscr{S}_{x_0 - \sigma t}\varphi\|_{H^1} + \|v(\cdot, t) - \mathscr{S}_{x_0 - \sigma t}\psi\|_{H^1} \to 0 \qquad \text{as} \quad t \to \infty$$

whenever

$$\|z(\cdot, 0)\|_{H^3} + \|w(\cdot, 0)\|_{H^2} \leqslant \varepsilon.$$

It will be useful to review the basic steps in deriving Theorem 2.1 above. To simplify the notation, let us write (φ, ψ) for $(\mathscr{S}_{x_0}\varphi, \mathscr{S}_{x_0}\psi)$, that is, we translate $(\varphi, \psi)(x)$ to $(\varphi, \psi)(x + x_0)$ so that we assume, for simplicity,

$$\int_{-\infty}^{+\infty} (u(x, t) - \varphi(x - \sigma t))\, dx = 0.$$

Now, both (u, v) and (φ, ψ) satisfy (1); by taking the difference of the two systems of equations and integrating the first equation with respect to x, we obtain

$$z_t + f(z_x + \varphi, w + \psi) - f(\varphi, \psi) = 0 \tag{15a}$$

$$w_t + (g(z_x + \varphi, w + \psi) - g(\varphi, \psi))_x = h(z_x + \varphi, w + \psi) - h(\varphi, \psi) \tag{15b}$$

In view of condition (4), we get from (15a)

$$w = -(z_t + f_u(\varphi + \theta_1 z_x, \psi + \theta_2 w) z_x)/f_v(\varphi + \theta_1 z_x, \psi + \theta_2 w) \tag{16a}$$

$$w = -(z_t + f_u(\varphi, \psi) z_x + \mathscr{Q}_1(f))/f_v(\varphi, \psi) \tag{16b}$$

where $0 < \theta_1, \theta_2 < 1$ and, for any $\mathscr{F}(u, v)$,

$$\mathscr{Q}_1(\mathscr{F}) \equiv \mathscr{F}(z_x + \varphi, w + \psi) - \mathscr{F}(\varphi, \psi) - \mathscr{F}_u(\varphi, \psi) z_x - \mathscr{F}_v(\varphi, \psi) w = O(1)(z_x^2 + w^2). \tag{17}$$

To simplify the notation, whenever a function is evaluated at (φ, ψ), we will omit the argument; thus, $g_u \equiv g_u(\varphi, \psi)$, $g_x \equiv g(\varphi, \psi)_x$, and so on. Plugging (16b) into (15b), we get, after multiplying by f_v and rearranging a few terms,

$$z_{tt} + (\lambda_1 + \lambda_2) z_{xt} + \lambda_1 \lambda_2 z_{xx} - h_v \cdot (z_t + \mu z_x) = \tilde{\mathscr{R}}(x, t) \tag{18}$$

where

$$\begin{aligned}\tilde{\mathscr{R}}(x,t) \equiv & (f_v^{-1} f_{vt} + f_v^{-1} g_v f_{vx} - g_{vx})(z_t + f_u z_x) \\ & + \mathscr{Q}_1(f)(f_v^{-1} f_{vt} - g_{vx} + g_v f_v^{-1} f_{vx} + h_v) \\ & + (f_v g_{ux} - f_{ut} - g_v f_{ux}) z_x - f_v \mathscr{Q}_1(h) \\ & + f_v \mathscr{Q}_1(g)_x - \mathscr{Q}_1(f)_t - g_v \mathscr{Q}_1(f)_x \end{aligned} \tag{19}$$

and

$$\mu \equiv f_u - f_v h_u h_v^{-1} \tag{20}$$

where, as noted above, λ_1, λ_2, μ, h_v, etc., are all evaluated at (φ, ψ).

We observe from (19) that the right-hand side $\tilde{\mathscr{R}}$ in (18) involves only high powers of terms which we expect to be small, so that it will probably have little effect in the overall analysis. All linear powers are written on the left-hand side in (18), which contains a first-order term with speed μ given in (20); this is the *dynamic* characteristic speed governing the propagation of weak disturbances over the traveling wave. We recall that the *equilibrium* speed $\lambda_*(u)$ given in (7) can be written as $\mu(u, v_*(u))$, so that for a non-equilibrium regime (u, v) like the traveling wave (φ, ψ) they will be different in general.

As a result of the nonlinearities present in the system (1), this characteristic speed μ changes along the wave. A simple computation shows that

$$\mu_x = (f_*''(u_{\text{avg}}) + O(|u_+ - u_-|))\,\varphi_x \tag{21}$$

where $u_{\text{avg}} \equiv \frac{1}{2}(u_+ + u_-)$. The simplest nonlinearity occurs when μ changes monotonically, and we will assume accordingly that $f_*(u) = f(u, v_*(u))$ is convex, as in the case of the traffic flow and other problems discussed in Whitham [10] (see (5c) above). In this case, we have $u_- > u_+$, with φ_x negative at every finite point; for sufficiently weak traveling waves (φ, ψ), i.e., $|u_+ - u_-|$ small, (21) then gives

$$\mu_x < 0 \tag{22}$$

and

$$M^{-1}(|\varphi_x| + |\psi_x|) \leqslant |\mu_x| \leqslant M\,|\varphi_x| \qquad \text{for all } x, t \tag{23}$$

for some positive constant M.

Also, for weak waves (φ, ψ),

$$|\mu - \sigma| = O(|u_+ - u_-|) \ll 1 \tag{24}$$

and, observing that $\varphi_x = O(|u_+ - u_-|)$,

$$|\varphi_x| + |\psi_x| + |\mu_x| = O(|u_+ - u_-|) \ll 1.$$

It is convenient to perform the analysis in a reference frame sitting on the traveling wave. Thus, we introduce the new variables (ξ, t), $\xi = x - \sigma t$, in terms of which (18) reads as

$$\begin{aligned} z_{tt} + (\lambda_1 + \lambda_2 - 2\sigma) z_{\xi t} + (\sigma - \lambda_1)(\sigma - \lambda_2) z_{\xi\xi} \\ + h_v(\sigma - \mu) z_\xi - h_v z_t = \mathscr{R}(\xi, t) \end{aligned} \tag{25}$$

where

$$\begin{aligned} \mathscr{R}(\xi, t) = {} & (f_v^{-1} g_v f_{v\xi} - \sigma f_v^{-1} f_{v\xi} - g_{v\xi})(z_t - \sigma z_\xi + f_u z_\xi) \\ & + \mathscr{Q}_1(f)(h_v + g_v f_v^{-1} f_{v\xi} - g_{v\xi} - \sigma f_v^{-1} f_{v\xi}) \\ & + (f_v g_{u\xi} + \sigma f_{u\xi} - g_v f_{u\xi}) z_\xi + f_v \mathscr{Q}_1(g)_\xi \\ & + \sigma \mathscr{Q}_1(f)_\xi - f_v \mathscr{Q}_1(h) - \mathscr{Q}_1(f)_t - g_v \mathscr{Q}_1(f)_\xi \end{aligned} \tag{26}$$

As noted in [5], the stability of the traveling wave is a consequence of three basic facts: its *compressibility*, expressed by (22) or, in the new variables (ξ, t), by the inequality $\mu_\xi < 0$; the fact that the wave travels at *subcharacteristic speeds*, see (9), which makes $(\sigma - \lambda_1)(\lambda_2 - \sigma)$ bounded below from zero; and the *relaxation effects* introduced by the rate term h in (1b), see (2), (3). In fact, under the assumptions outlined above, Liu [5] was able to derive, after considerable work, the energy estimate

$$\begin{aligned} & \int_{-\infty}^{+\infty} \left(\sum_{j=0}^{3} |\nabla^j z|^2(x, t) + \sum_{j=0}^{2} |\nabla^j w|^2 (x, t) \right) dx \\ & \quad + \int_0^T \int_{-\infty}^{+\infty} \left(|\mu_x| z^2 + \sum_{j=1}^{3} |\nabla^j z|^2 + \sum_{j=0}^{2} |\nabla^j w|^2 \right) dx\, dt \\ & = O(1)\ \left(\|z(\cdot, 0)\|_{H^3}^2 + \|w(\cdot, 0)\|_{H^2}^2 \right) \end{aligned} \tag{27}$$

provided we take $\|z(\cdot, 0)\|_{H^3}^2 + \|w(\cdot, 0)\|_{H^2}^2$ sufficiently small, see [5] for the detailed derivation. We simply remark that (27) immediately gives

$$\sum_{j=0}^{2} \|\nabla^j z(\cdot, t)\|_{L^\infty} + \sum_{j=0}^{1} \|\nabla^j w(\cdot, t)\|_{L^\infty} \to 0 \qquad \text{as} \quad t \to +\infty,$$

but, due to the fact that $\mu_x \to 0$ as $x \to \pm\infty$, no decay rate can be directly inferred from (27). This will be done by a different, though related, analysis in the next section.

3. DECAY RATE ANALYSIS

We will now improve the stability results discussed in Section 2 above by deriving decay estimates for the disturbances (z, w) over the traveling wave (φ, ψ) introduced by the initial data (10). It turns out that these time decay rates depend upon how fast the initial disturbances die off on the real line. Thus, besides the conditions (11a), (11b) of Theorem 2.1, we assume that $(\bar{u}, \bar{v})$ is such that

$$\int_{-\infty}^{+\infty} (1+|x|^N)(|\bar{u}(x)|^2 + |\bar{v}(x)|^2)\ dx \tag{28a}$$

and

$$\int_{-\infty}^{0} (1+|x|^N) \left| \int_{-\infty}^{x} \bar{u}(y)\ dy \right|^2 dx \ + \int_{0}^{+\infty} (1+|x|^N) \left| \int_{x}^{+\infty} \bar{u}(y)\ dy \right|^2 dx \tag{28b}$$

are both finite for some positive integer N, which is certainly the case when $\bar{u}(x)$, $\bar{v}(x)$ decay algebraically fast as $x \to \pm\infty$. This allows us to show

THEOREM 3.1. *Under all the hypotheses of Theorem* 2.1, *assume furthermore that for the initial disturbance* $(\bar{u}, \bar{v})$ *both* (28a) *and* (28b) *are finite for some positive integer N. Then there exist positive constants* δ, ε *such that*

$$\|(u, v)(\cdot, t) - (\mathcal{S}_{x_0 - \sigma t}\varphi, \mathcal{S}_{x_0 - \sigma t}\psi)\|_{H^2} = O(1)(1+t)^{-N/2}$$

whenever

$$|u_+ - u_-| \leqslant \delta \qquad \textit{and} \qquad \|z(\cdot, 0)\|_{H^3} + \|w(\cdot, 0)\|_{H^2} \leqslant \varepsilon.$$

Before we prove Theorem 3.1, we will present the basic reasoning behind the argument. The key step is to strip off the dynamics for the disturbances described by equations (25), (26) to its barest form. We do this in the following way. To a first approximation, we can think of equation (25) as

$$z_t + (\mu - \sigma) z_\xi = 0 \tag{29}$$

where $\mu = \mu(\varphi(\xi), \psi(\xi))$ is a function of ξ only and σ is a constant, the propagation speed of the traveling wave (φ, ψ), see (8), (20). A simple computation gives [5]

$$\mu(\xi) - \sigma = \frac{(\lambda_1 - \sigma)(\lambda_2 - \sigma)}{h h_v} h_\xi;$$

since $h(\varphi(\xi), \psi(\xi))$ must vanish at the equilibrium states at $\xi = \pm\infty$, we then get

$$\mu(\xi) - \sigma = 0 \qquad \text{for some} \quad \xi_0 \in \mathbf{R} \tag{30}$$

Thus, recalling (22), at large times most of the information for solutions of (29) come from points ξ far away from ξ_0 on the initial line. This suggests that we consider a decay factor of the form $t^\alpha |\xi - \xi_0|^\beta$, involving both t and ξ, in deriving energy inequalities with time decay for (25), or, to avoid the singular behavior of the absolute value at zero, $(1+t)^\alpha \langle \xi - \xi_0 \rangle^\beta$, where

$$\langle \xi \rangle \equiv \sqrt{1 + \xi^2} \tag{31}$$

α and β being non-negative constants which are at our disposal. In what follows here we assume that we have, in view of (27),

$$\| z(\,\cdot\,, t) \|_{H^3} + \| w(\,\cdot\,, t) \|_{H^2} \leqslant \varepsilon \qquad \text{for all} \quad t \geqslant 0 \tag{32}$$

where $\varepsilon \ll 1$, and similarly,

$$|u_+ - u_-| = \delta, \qquad \delta \text{ small} \tag{33}$$

Using (2), (22) and (30), it is straightforward to show [11]

$$\begin{aligned}
&-\int_{-\infty}^{+\infty} \langle \xi - \xi_0 \rangle^\beta \, h_v \cdot (\mu - \sigma)\, z z_\xi \, d\xi \, dt \\
&= \tfrac{1}{4} \int_{-\infty}^{+\infty} \langle \xi - \xi_0 \rangle^\beta \, |h_v| \cdot |\mu_\xi| \, z^2 \, d\xi \\
&\quad + \tfrac{1}{2} \int_{-\infty}^{+\infty} \langle \xi - \xi_0 \rangle^\beta \, h_{v\xi} \cdot (\mu - \sigma) z^2 \, d\xi \\
&\quad + \tfrac{1}{2} \int_{-\infty}^{+\infty} \langle \xi - \xi_0 \rangle^{\beta - 1} \, \mathscr{M}(\xi) z^2 \, d\xi
\end{aligned} \tag{34}$$

where

$$\mathscr{M}(\xi) \equiv \tfrac{1}{2} |h_v| \cdot |\mu_\xi| \, \langle \xi - \xi_0 \rangle + \beta \, \frac{|\xi - \xi_0|}{\langle \xi - \xi_0 \rangle} \, |\mu - \sigma|.$$

It is important to note that we have

$$\mathscr{M}(\xi) \geqslant m \cdot \beta \qquad \text{for all} \quad \xi \in \mathbf{R} \tag{35}$$

for some positive constant $m = m(\delta)$, which will be used below. We now get to the most important step of the whole analysis. Taking $C \gg 1$, multiplying (25) by $(1+T)^\alpha \langle \xi - \xi_0 \rangle^\beta \ (z/C + z_t)$ and integrating the result over $\mathbf{R} \times [0, T]$, one can show that, for ε and δ small,

$$\begin{aligned}
&(1+T)^\alpha \int_{-\infty}^{+\infty} \langle \xi \rangle^\beta (z^2(\xi, T) + z_\xi^2(\xi, T) + z_t^2(\xi, T))\, d\xi \\
&\quad + \int_0^T (1+t)^\alpha \int_{-\infty}^{+\infty} \langle \xi \rangle^\beta (|\mu_\xi|\, z^2 + z_\xi^2 + z_t^2)\, d\xi\, dt \\
&\quad + \beta \int_0^T (1+t)^\alpha \int_{-\infty}^{+\infty} \langle \xi \rangle^{\beta-1} z^2\, d\xi\, dt \\
&= O(1) \left\{ \int_{-\infty}^{+\infty} \langle \xi \rangle^\beta (z^2(\xi, 0) + z_\xi^2(\xi, 0) + z_t^2(\xi, 0))\, d\xi \right. \\
&\quad + \alpha \int_0^T (1+t)^{\alpha-1} \int_{-\infty}^{+\infty} \langle \xi \rangle^\beta (z^2 + z_\xi^2 + z_t^2)\, d\xi\, dt \\
&\quad \left. + \beta \int_0^T (1+t)^\alpha \int_{-\infty}^{+\infty} (z_\xi^2 + z_t^2)\, d\xi\, dt \right\}.
\end{aligned} \tag{36}$$

This weighted energy inequality will be the key point in the decay analysis below. A detailed derivation is given in [11], but for completeness we will sketch the basic steps leading to (36). Multplying (25) by $(1+t)^\alpha \langle \xi - \xi_0 \rangle^\beta z$ and integrating the result over $\mathbf{R} \times [0, T]$, we get, after a few integrations by parts,

$$\begin{aligned}
&\tfrac{1}{2}(1+T)^\alpha \int_{-\infty}^{+\infty} \langle \xi - \xi_0 \rangle^\beta |h_v|\, z^2(\xi, T)\, d\xi \\
&\quad - \int_0^T (1+t)^\alpha \int_{-\infty}^{+\infty} \langle \xi - \xi_0 \rangle^\beta h_v \cdot (\mu - \sigma)\, z z_\xi\, d\xi\, dt \\
&\quad + \int_0^T (1+t)^\alpha \int_{-\infty}^{+\infty} |\lambda_1 - \sigma| \cdot |\lambda_2 - \sigma|\, \langle \xi - \xi_0 \rangle^\beta z_\xi^2\, d\xi\, dt
\end{aligned}$$

$$= -(1+T)^{\alpha} \int_{-\infty}^{+\infty} \langle \xi - \xi_0 \rangle^{\beta} z(\xi, T)\, z_t(\xi, T)\, d\xi$$

$$+ \tfrac{1}{2} \int_{-\infty}^{+\infty} \langle \xi - \xi_0 \rangle^{\beta} |h_v| z^2 (\xi, 0)\, d\xi$$

$$+ \int_{-\infty}^{+\infty} \langle \xi - \xi_0 \rangle^{\beta} z(\xi,0)\, z_t(\xi, 0)\, d\xi$$

$$+ \alpha \int_0^T (1+t)^{\alpha-1} \int_{-\infty}^{+\infty} \langle \xi - \xi_0 \rangle^{\beta} z z_t\, d\xi\, dt$$

$$+ \int_0^T (1+t)^{\alpha} \int_{-\infty}^{+\infty} \langle \xi - \xi_0 \rangle^{\beta} (\lambda_1 + \lambda_2 - 2\sigma)\, z_{\xi} z_t\, d\xi\, dt$$

$$+ \int_0^T (1+t)^{\alpha} \int_{-\infty}^{+\infty} \langle \xi - \xi_0 \rangle^{\beta} ((\lambda_1 + \lambda_2)_{\xi}\, z z_t + z_t^2)\, d\xi\, dt$$

$$+ \tfrac{1}{2}\alpha \int_0^T (1+t)^{\alpha-1} \int_{-\infty}^{+\infty} \langle \xi - \xi_0 \rangle^{\beta} |h_v|\, z^2\, d\xi\, dt$$

$$+ \beta \int_0^T (1+t)^{\alpha} \int_{-\infty}^{+\infty} \langle \xi - \xi_0 \rangle^{\beta-2} (\xi - \xi_0)(\lambda_1 + \lambda_2 - 2\sigma)\, z z_t\, d\xi\, dt$$

$$+ \beta \int_0^T (1+t)^{\alpha} \int_{-\infty}^{+\infty} \langle \xi - \xi_0 \rangle^{\beta-2} (\xi - \xi_0)(\lambda_1 - \sigma)(\lambda_2 - \sigma)\, z z_{\xi}\, d\xi\, dt$$

$$+ \int_0^T (1+t)^{\alpha} \int_{-\infty}^{+\infty} \langle \xi - \xi_0 \rangle^{\beta} ((\lambda_1 - \sigma)(\lambda_2 - \sigma))_{\xi}\, z z_{\xi}\, d\xi\, dt$$

$$+ \int_0^T (1+t)^{\alpha} \int_{-\infty}^{+\infty} \langle \xi - \xi_0 \rangle^{\beta} \mathscr{R}(\xi, t)\, d\xi\, dt \tag{37}$$

where we have used (2), (9), (22). From (4), (9), (23), (34) and (35), we can rewrite (37) as

$$(1+T)^{\alpha} \int_{-\infty}^{+\infty} \langle \xi - \xi_0 \rangle^{\beta} z^2(\xi, T)\, d\xi$$

$$+ \int_0^T (1+t)^{\alpha} \int_{-\infty}^{+\infty} \langle \xi - \xi_0 \rangle^{\beta} z_{\xi}^2\, d\xi\, dt$$

$$+ \int_0^T (1+t)^{\alpha} \int_{-\infty}^{+\infty} \langle \xi - \xi_0 \rangle^{\beta} |\mu_{\xi}|\, z^2\, d\xi\, dt$$

$$+ m \cdot \beta \int_0^T (1+t)^{\alpha} \int_{-\infty}^{+\infty} \langle \xi - \xi_0 \rangle^{\beta-1} z^2\, d\xi\, dt$$

$$
\begin{aligned}
&= O(1) \left\{ \int_{-\infty}^{+\infty} \langle \xi - \xi_0 \rangle^\beta \left(z^2(\xi, 0) + z_t^2(\xi, 0) \right) d\xi \right. \\
&\quad + (1+T)^\alpha \int_{-\infty}^{+\infty} \langle \xi - \xi_0 \rangle^\beta \, z_t^2(\xi, T) \, d\xi \\
&\quad + \int_0^T (1+t)^\alpha \int_{-\infty}^{+\infty} \langle \xi - \xi_0 \rangle^\beta \, z_t^2 \, d\xi \, dt \\
&\quad + \alpha \int_0^T (1+t)^{\alpha-1} \int_{-\infty}^{+\infty} \langle \xi - \xi_0 \rangle^\beta \left(z^2 + z_t^2 \right) d\xi \, dt \\
&\quad + \beta \int_0^T (1+t)^\alpha \int_{-\infty}^{+\infty} \langle \xi - \xi_0 \rangle^{\beta-1} \left(|z_\xi| + |z_t| \right) |z| \, d\xi \, dt \\
&\quad \left. + \left| \int_0^T (1+t)^\alpha \int_{-\infty}^{+\infty} \langle \xi - \xi_0 \rangle^\beta \, z \mathscr{R}(\xi, t) \, d\xi \, dt \right| \right\} \qquad (38)
\end{aligned}
$$

We must now estimate the last two terms on the right-hand side of (38). We have, for a given $\varepsilon > 0$,

$$
\begin{aligned}
&\int_{-\infty}^{+\infty} \langle \xi - \xi_0 \rangle^{\beta-1} \left(|z_\xi| + |z_t| \right) |z| \, d\xi \\
&\qquad \leqslant \varepsilon m \int_{-\infty}^{+\infty} \langle \xi - \xi_0 \rangle^{\beta-1} z^2 \, d\xi + \varepsilon \int_{-\infty}^{+\infty} \langle \xi - \xi_0 \rangle^\beta \left(z_\xi^2 + z_t^2 \right) d\xi \\
&\qquad\quad + \mathscr{K}(\delta) \int_{-\infty}^{+\infty} \left(z_\xi^2 + z_t^2 \right) d\xi \qquad (39)
\end{aligned}
$$

where $\mathscr{K}(\delta)$ is a positive constant depending on δ, see (33), and $m > 0$ is given in (35) above. As to the last term in (38), we can show, using (16), (17), (23), (32), (33), after a few computations,

$$
\begin{aligned}
&\int_0^T (1+t)^\alpha \int_{-\infty}^{+\infty} \langle \xi - \xi_0 \rangle^\beta \, z \mathscr{R}(\xi, t) \, d\xi \, dt \\
&\quad = O(1)(\sqrt{\delta} + \varepsilon) \int_0^T (1+t)^\alpha \int_{-\infty}^{+\infty} \langle \xi - \xi_0 \rangle^\beta \left(|\mu_\xi| \, z^2 + z_\xi^2 + z_t^2 \right) d\xi \, dt \\
&\qquad + O(\varepsilon)(1+T)^\alpha \int_{-\infty}^{+\infty} \langle \xi - \xi_0 \rangle^\beta \left(z^2(\xi, T) + z_\xi^2(\xi, T) + z_t^2(\xi, T) \right. d\xi \\
&\qquad + O(\varepsilon) \int_{-\infty}^{+\infty} \langle \xi - \xi_0 \rangle^\beta \left(z^2(\xi, 0) + z_\xi^2(\xi, 0) + z_t^2(\xi, 0) \right. d\xi \qquad (40)
\end{aligned}
$$

Taking (39) and (40) into (38), we obtain, for ε and δ small,

$$
\begin{aligned}
&(1+T)^{\alpha} \int_{-\infty}^{+\infty} \langle \xi \rangle^{\beta} z^2(\xi, T)\, d\xi + \int_0^T (1+t)^{\alpha} \int_{-\infty}^{+\infty} \langle \xi \rangle^{\beta} z_{\xi}^2 \, d\xi\, dt \\
&\quad + \int_0^T (1+t)^{\alpha} \int_{-\infty}^{+\infty} \langle \xi \rangle^{\beta} |\mu_{\xi}|\, z^2 \, d\xi\, dt \\
&\quad + m \cdot \beta \int_0^T (1+t)^{\alpha} \int_{-\infty}^{+\infty} \langle \xi \rangle^{\beta-1} z^2 \, d\xi\, dt \\
&\quad = O(1) \left\{ \int_{-\infty}^{+\infty} \langle \xi \rangle^{\beta} (z^2(\xi, 0) + z_{\xi}^2(\xi, 0) + z_t^2(\xi, 0)\, d\xi \right. \\
&\quad + \alpha \int_0^T (1+t)^{\alpha-1} \int_{-\infty}^{+\infty} \langle \xi \rangle^{\beta} (z^2 + z_t^2)\, d\xi\, dt \\
&\quad + \mathscr{K}(\delta) \beta \int_0^T (1+t)^{\alpha} \int_{-\infty}^{+\infty} (z_{\xi}^2 + z_t^2)\, d\xi\, dt \\
&\quad + \int_0^T (1+t)^{\alpha} \int_{-\infty}^{+\infty} \langle \xi \rangle^{\beta} z_t^2 \, d\xi\, dt \\
&\quad \left. + (1+T)^{\alpha} \int_{-\infty}^{+\infty} \langle \xi \rangle^{\beta} (z_t^2(\xi, T) + \varepsilon z_{\xi}^2(\xi, T))\, d\xi \right\}. \qquad (41)
\end{aligned}
$$

To make (41) useful, we have to estimate the last two terms on the right-hand side. To this end, we multiply (25) by $(1+t)^{\alpha} \langle \xi - \xi_0 \rangle^{\beta} z_t$ and integrate the result in $\mathbf{R} \times [0, T]$, which gives, after a few computations,

$$
\begin{aligned}
&\tfrac{1}{2}(1+T)^{\alpha} \int_{-\infty}^{+\infty} \langle \xi \rangle^{\beta} (z_t^2(\xi, T) + |\lambda_1 - \sigma| \cdot |\lambda_2 - \sigma|\, z_{\xi}^2(\xi, T)\, d\xi \\
&\quad + \int_0^T (1+t)^{\alpha} \int_{-\infty}^{+\infty} |h_v|\, \langle \xi \rangle^{\beta} z_t^2 \, d\xi\, dt \\
&\quad = \tfrac{1}{2} \int_{-\infty}^{+\infty} \langle \xi \rangle^{\beta} (z_t^2(\xi, 0) + |\lambda_1 - \sigma| \cdot |\lambda_2 - \sigma|\, z_{\xi}^2(\xi, 0)\, d\xi \\
&\quad - \int_0^T (1+t)^{\alpha} \int_{-\infty}^{+\infty} \langle \xi \rangle^{\beta} (\lambda_1 + \lambda_2 - 2\sigma)\, z_t z_{\xi t}\, d\xi\, dt \\
&\quad + \frac{\alpha}{2} \int_0^T (1+t)^{\alpha-1} \int_{-\infty}^{+\infty} \langle \xi \rangle^{\beta} (z_t^2 + |\lambda_1 - \sigma| \cdot |\lambda_2 - \sigma|\, z_{\xi}^2)\, d\xi\, dt
\end{aligned}
$$

$$+\int_0^T (1+t)^\alpha \int_{-\infty}^{+\infty} (\langle \xi \rangle^\beta (\lambda_1-\sigma)(\lambda_2-\sigma))_\xi \, z_\xi z_t \, d\xi \, dt$$

$$+\int_0^T (1+t)^\alpha \int_{-\infty}^{+\infty} h_v \langle \xi \rangle^\beta (\mu-\sigma) z_\xi z_t \, d\xi \, dt$$

$$+\int_0^T (1+t)^\alpha \int_{-\infty}^{+\infty} \langle \xi \rangle^\beta z_t \mathscr{R} \, d\xi \, dt \tag{42}$$

in view of (2), (9). On the other hand, using (16), (17), (23), (33) it is not hard to get [11]

$$\int_0^T (1+t)^\alpha \int_{-\infty}^{+\infty} \langle \xi \rangle^\beta z_t \mathscr{R}(\xi, t) \; d\xi \, dt$$

$$= (\delta+\varepsilon) \, O(1) \int_0^T (1+t)^\alpha \int_{-\infty}^{+\infty} \langle \xi \rangle^\beta (z_t^2 + z_\xi^2) \; d\xi \, dt \tag{43}$$

Using (24), (33), (43), we then get from (42) that, for ε and δ small,

$$(1+T)^\alpha \int_{-\infty}^{+\infty} \langle \xi \rangle^\beta \; (z_t^2(\xi, T) + z_\xi^2(\xi, T)) \, d\xi$$

$$+\int_0^T (1+t)^\alpha \int_{-\infty}^{+\infty} \langle \xi \rangle^\beta z_t^2 \, d\xi \, dt$$

$$= O(1) \; \Big\{ \int_{-\infty}^{+\infty} \langle \xi \rangle^\beta (z_\xi^2(\xi, 0) + z_t^2(\xi, 0) \, d\xi$$

$$+ (\varepsilon+\delta) \int_0^T (1+t)^\alpha \int_{-\infty}^{+\infty} \langle \xi \rangle^\beta z_\xi^2 \, d\xi \, dt$$

$$+ \beta \int_0^T (1+t)^\alpha \int_{-\infty}^{+\infty} \langle \xi \rangle^{\beta-1} (z_t^2 + z_\xi^2) \; d\xi \, dt$$

$$+ \alpha \int_0^T (1+t)^{\alpha-1} \int_{-\infty}^{+\infty} \langle \xi \rangle^\beta (z_t^2 + z_\xi^2) \, d\xi \, dt \Big\}. \tag{44}$$

Now,

$$\int_{-\infty}^{+\infty} \langle \xi \rangle^{\beta-1} (z_t^2 + z_\xi^2) \, d\xi$$

$$\leqslant \mathscr{L}(\varepsilon) \int_{-\infty}^{+\infty} (z_t^2 + z_\xi^2) \, d\xi + \varepsilon \int_{-\infty}^{+\infty} \langle \xi \rangle^\beta (z_t^2 + z_\xi^2) \, d\xi$$

for some positive constant $\mathscr{L}(\varepsilon)$ which depends on ε, so that (44) can be rewritten as

$$
\begin{aligned}
&(1+T)^{\alpha}\int_{-\infty}^{+\infty}\langle\xi\rangle^{\beta}\,(z_t^2(\xi,T)+z_\xi^2(\xi,T))\,d\xi\\
&\quad+\int_0^T(1+t)^{\alpha}\int_{-\infty}^{+\infty}\langle\xi\rangle^{\beta}\,z_t^2\,d\xi\,dt\\
&\quad=O(1)\left\{\int_{-\infty}^{+\infty}\langle\xi\rangle^{\beta}\,(z_\xi^2(\xi,0)+z_t^2(\xi,0))\,d\xi\right.\\
&\qquad+(\varepsilon+\delta)\int_0^T(1+t)^{\alpha}\int_{-\infty}^{+\infty}\langle\xi\rangle^{\beta}\,z_\xi^2\,d\xi\,dt\\
&\qquad+\mathscr{L}(\varepsilon)\beta\int_0^T(1+t)^{\alpha}\int_{-\infty}^{+\infty}(z_t^2+z_\xi^2)\,d\xi\,dt\\
&\qquad\left.+\alpha\int_0^T(1+t)^{\alpha-1}\int_{-\infty}^{+\infty}\langle\xi\rangle^{\beta}\,(z_t^2+z_\xi^2)\,d\xi\,dt\right\}.
\end{aligned}
\tag{45}
$$

Finally, choosing $C>0$ big enough, we multiply (41) by $1/C$ and add the result to (45) to get (36) for ε and δ sufficiently small.

Having derived (36), we are in a good position to proceed. We first rewrite (36) in the following equivalent way, using (4), (15), (16):

$$
\begin{aligned}
&(1+T)^{\alpha}\int_{-\infty}^{+\infty}\langle\xi\rangle^{\beta}\,(z^2(\xi,T)+z_\xi^2(\xi,T)+z_t^2(\xi,T)+w^2(\xi,T))\,d\xi\\
&\quad+\int_0^T(1+t)^{\alpha}\int_{-\infty}^{+\infty}\langle\xi\rangle^{\beta}\,(|\mu_\xi|z^2+z_\xi^2+z_t^2+w^2)\,d\xi\,dt\\
&\quad+\beta\int_0^T(1+t)^{\alpha}\int_{-\infty}^{+\infty}\langle\xi\rangle^{\beta-1}\,z^2\,d\xi\,dt\\
&=O(1)\left\{\int_{-\infty}^{+\infty}\langle\xi\rangle^{\beta}\,(z^2(\xi,0)+z_\xi^2(\xi,0)+w^2(\xi,0))\,d\xi\right.\\
&\quad+\alpha\int_0^T(1+t)^{\alpha-1}\int_{-\infty}^{+\infty}\langle\xi\rangle^{\beta}\,(z^2+z_\xi^2+z_t^2)\,d\xi\,dt\\
&\quad\left.+\beta\int_0^T(1+t)^{\alpha}\int_{-\infty}^{+\infty}(z_\xi^2+z_t^2)\,d\xi\,dt\right\}.
\end{aligned}
\tag{46}
$$

We now proceed in much the same way as in Kawashima and Matsumura [2]. Having assumed that (28a) and (28b) are both finite for a certain integer $N \geqslant 1$, this immediately gives

$$\int_{-\infty}^{+\infty} \langle \xi \rangle^N (z^2(\xi, 0) + z_\xi^2(\xi, 0) + w^2(\xi, 0))\, d\xi < \infty \tag{47}$$

where $\langle \xi \rangle$ is defined in (31). Then, taking $\beta = N, \alpha = 0$ in (46), we immediately get

$$\int_{-\infty}^{+\infty} \langle \xi \rangle^N (z^2(\xi, T) + z_\xi^2(\xi, T) + z_t^2(\xi, T) + w^2(\xi, T))\, d\xi$$
$$+ \int_0^T \int_{-\infty}^{+\infty} \langle \xi \rangle^N (|\mu_\xi|\, z^2 + z_\xi^2 + z_t^2 + w^2)\, d\xi\, dt$$
$$+ \int_0^T \int_{-\infty}^{+\infty} \langle \xi \rangle^{N-1} z^2\, d\xi\, dt = O(1).$$

Hence, taking now $\beta = 0, \alpha = 1$, one gets the estimate

$$(1+T) \int_{-\infty}^{+\infty} (z^2(\xi, T) + z_\xi^2(\xi, T) + z_t^2(\xi, T) + w^2(\xi, T))\, d\xi$$
$$+ \int_0^T (1+t) \int_{-\infty}^{+\infty} (|\mu_\xi| z^2 + z_\xi^2 + z_t^2 + w^2)\, d\xi\, dt = O(1), \tag{48}$$

which in turn allows us to consider $\beta = N-1$, $\alpha = 1$ in (46) above, thus improving (48) to

$$(1+T) \int_{-\infty}^{+\infty} \langle \xi \rangle^{N-1} (z^2(\xi, T) + z_\xi^2(\xi, T) + z_t^2(\xi, T) + w^2(\xi, T))\, d\xi$$
$$+ \int_0^T (1+t) \int_{-\infty}^{+\infty} \langle \xi \rangle^{N-1} (|\mu_\xi| z^2 + z_\xi^2 + z_t^2 + w^2)\, d\xi\, dt$$
$$+ \int_0^T (1+t) \int_{-\infty}^{+\infty} \langle \xi \rangle^{N-2} z^2\, d\xi\, dt = O(1).$$

Proceeding in this way, i.e., taking successively in (46) $\beta = N - j$, $\alpha = j$, and then $\beta = 0$, $\alpha = j+1$, for $j = 0, 1, 2, ..., N-1$, we end up with the following decay estimate

$$\sum_{j=0}^{N} (1+T)^j \int_{-\infty}^{+\infty} \langle \xi \rangle^{N-j} (z^2(\xi, T) + z_\xi^2(\xi, T) + z_t^2(\xi, T) + w^2(\xi, T))\, d\xi$$
$$+ \sum_{j=0}^{N} \int_0^T (1+t)^j \int_{-\infty}^{+\infty} \langle \xi \rangle^{N-j} (|\mu_\xi|\, z^2 + z_\xi^2 + z_t^2 + w^2)\, d\xi\, dt$$
$$+ \sum_{j=0}^{N-1} \int_0^T (1+t)^j \int_{-\infty}^{+\infty} \langle \xi \rangle^{N-1-j} z^2\, d\xi\, dt \leqslant \mathscr{C},$$

where $\mathscr{C}$ is a positive constant depending on the initial data but not on T. In particular, we have

$$(1+T)^N \int_{-\infty}^{+\infty} (z^2(\xi, T) + z_\xi^2(\xi, T) + z_t^2(\xi, T) + w^2(\xi, T))\, d\xi$$
$$+ \int_0^T (1+t)^N \int_{-\infty}^{+\infty} (|\mu_\xi|\, z^2 + z_\xi^2 + z_t^2 + w^2)\, d\xi\, dt$$
$$+ \int_0^T (1+t)^{N-1} \int_{-\infty}^{+\infty} z^2\, d\xi\, dt = O(1). \tag{49}$$

We can now proceed in a similar way as in Liu [5]. Differentiating (25) with respect to ξ, multiplying the resulting equation by $(1+t)^\alpha z_\xi$ and integrating over $\mathbf{R} \times [0, T]$, we get, after some straightforward work,

$$(1+T)^\alpha \int_{-\infty}^{+\infty} z_\xi^2(\xi, T)\, d\xi + \int_0^T (1+t)^\alpha \int_{-\infty}^{+\infty} z_{\xi\xi}^2\, d\xi\, dt$$
$$= O(1) \left\{ \int_{-\infty}^{+\infty} (z_\xi^2(\xi, 0) + z_{\xi t}^2(\xi, 0))\, d\xi + (1+T)^\alpha \int_{-\infty}^{+\infty} z_{\xi t}^2(\xi, T)\, d\xi \right.$$
$$+ \int_0^T (1+t)^\alpha \int_{-\infty}^{+\infty} (z_{tt}^2 + z_{\xi t}^2)\, d\xi\, dt$$
$$+ \alpha \int_0^T (1+t)^{\alpha-1} \int_{-\infty}^{+\infty} (z_\xi^2 + z_{\xi t}^2)\, d\xi\, dt$$
$$\left. + (\varepsilon + \delta) \int_0^T (1+t)^\alpha \int_{-\infty}^{+\infty} (z_\xi^2 + z_t^2 + z_{\xi\xi}^2 + z_{\xi t}^2)\, d\xi\, dt \right\}, \tag{50}$$

where we have used (32), (33). In a similar way, differentiating (25) with respect to t, multiplying the result by $(1+t)^\alpha z_{tt}$ and integrating over $\mathbf{R} \times [0, T]$, we can show, using (32), (33) again,

$$(1+T)^{\alpha} \int_{-\infty}^{+\infty} z_{tt}^2(\xi, T)\, d\xi + \int_0^T (1+t)^{\alpha} \int_{-\infty}^{+\infty} z_{tt}^2\, d\xi\, dt$$

$$= O(1) \left\{ \int_{-\infty}^{+\infty} z_{tt}^2(\xi, 0)\, d\xi + \int_0^T (1+t)^{\alpha} \int_{-\infty}^{+\infty} (z_{\xi\xi t}^2 + z_{\xi tt}^2)\, d\xi\, dt \right.$$

$$+ (\varepsilon + \delta) \int_0^T (1+t)^{\alpha} \int_{-\infty}^{+\infty} (z_{\xi}^2 + z_t^2 + z_{\xi\xi}^2 + z_{\xi t}^2 + z_{tt}^2 + z_{\xi\xi\xi}^2)\, d\xi\, dt$$

$$\left. + \alpha \int_0^T (1+t)^{\alpha-1} \int_{-\infty}^{+\infty} z_{tt}^2\, d\xi\, dt \right\}. \tag{51}$$

If we now differentiate (25) with respect to ξ, multiply the result by $(1+t)^{\alpha} z_{\xi t}$ and integrate over $\mathbf{R} \times [0, T]$, we get, after similar work,

$$(1+T)^{\alpha} \int_{-\infty}^{+\infty} z_{\xi t}^2(\xi, T)\, d\xi + \int_0^T (1+t)^{\alpha} \int_{-\infty}^{+\infty} z_{\xi t}^2\, d\xi\, dt$$

$$= O(1) \left\{ \int_{-\infty}^{+\infty} z_{\xi t}(\xi, 0)\, d\xi + \int_0^T (1+t)^{\alpha} \int_{-\infty}^{+\infty} z_{\xi\xi\xi}^2\, d\xi\, dt \right.$$

$$+ (\varepsilon + \delta) \int_0^T (1+t)^{\alpha} \int_{-\infty}^{+\infty} (z_{\xi}^2 + z_t^2 + z_{\xi\xi}^2 + z_{\xi t}^2)\, d\xi\, dt$$

$$\left. + \alpha \int_0^T (1+t)^{\alpha-1} \int_{-\infty}^{+\infty} z_{\xi t}^2\, d\xi\, dt \right\}. \tag{52}$$

Finally, integrating $(25)_{\xi\xi}(1+t)^{\alpha} z_{\xi\xi}$, $(25)_{\xi\xi}(1+t)^{\alpha} z_{\xi\xi t}$, $(25)_{\xi t}(1+t)^{\alpha} z_{\xi tt}$, and $(25)_{tt}(1+t)^{\alpha} z_{ttt}$ over $\mathbf{R} \times [0, T]$, we get, after a number of straightforward estimations,

$$(1+T)^{\alpha} \int_{-\infty}^{+\infty} z_{\xi\xi}^2(\xi, T)\, d\xi + \int_0^T (1+t)^{\alpha} \int_{-\infty}^{+\infty} z_{\xi\xi\xi}^2\, d\xi\, dt$$

$$= O(1) \left\{ (1+T)^{\alpha} \int_{-\infty}^{+\infty} z_{\xi\xi t}^2(\xi, T)\, d\xi + \int_{-\infty}^{+\infty} (z_{\xi\xi}^2(\xi, 0) + z_{\xi\xi t}^2(\xi, 0))\, d\xi \right.$$

$$+ \int_0^T (1+t)^{\alpha} \int_{-\infty}^{+\infty} z_{\xi\xi t}^2\, d\xi\, dt + \alpha \int_0^T (1+t)^{\alpha-1} \int_{-\infty}^{+\infty} (z_{\xi\xi}^2 + z_{\xi\xi t}^2)\, d\xi\, dt$$

$$\left. + (\varepsilon + \delta) \int_0^T (1+t)^{\alpha} \int_{-\infty}^{+\infty} (z_{\xi}^2 + z_t^2 + z_{\xi\xi}^2 + z_{\xi t}^2)\, d\xi\, dt \right\}, \tag{53}$$

$$
\begin{aligned}
&(1+T)^{\alpha} \int_{-\infty}^{+\infty} (z_{\xi\xi t}^2(\xi, T) + z_{\xi\xi\xi}^2(\xi, T))\, d\xi + \int_0^T (1+t)^{\alpha} \int_{-\infty}^{+\infty} z_{\xi\xi t}^2 \, d\xi\, dt \\
&\quad = O(1) \left\{ \int_{-\infty}^{+\infty} (z_{\xi\xi t}^2(\xi, 0) + z_{\xi\xi\xi}^2(\xi, 0))\, d\xi \right. \\
&\qquad + (\varepsilon + \delta) \int_0^T (1+t)^{\alpha} \\
&\qquad \times \int_{-\infty}^{+\infty} (z_{\xi}^2 + z_t^2 + z_{\xi\xi}^2 + z_{\xi t}^2 + z_{tt}^2 + z_{\xi\xi\xi}^2 + z_{\xi\xi t}^2 + z_{\xi tt}^2)\, d\xi\, dt \\
&\qquad \left. + \alpha \int_0^T (1+t)^{\alpha-1} \int_{-\infty}^{+\infty} (z_{\xi\xi\xi}^2 + z_{\xi\xi t}^2)\, d\xi\, dt \right\},
\end{aligned} \tag{54}
$$

$$
\begin{aligned}
&(1+T)^{\alpha} \int_{-\infty}^{+\infty} (z_{\xi tt}^2(\xi, T) + z_{\xi\xi t}^2(\xi, T))\, d\xi + \int_0^T (1+t)^{\alpha} \int_{-\infty}^{+\infty} z_{\xi tt}^2 \, d\xi\, dt \\
&\quad = O(1) \left\{ \int_{-\infty}^{+\infty} (z_{\xi tt}^2(\xi, 0) + z_{\xi\xi t}^2(\xi, 0))\, d\xi \right. \\
&\qquad + (\varepsilon + \delta) \int_0^T (1+t)^{\alpha} \\
&\qquad \times \int_{-\infty}^{+\infty} (z_{\xi}^2 + z_t^2 + z_{\xi\xi}^2 + z_{\xi t}^2 + z_{tt}^2 + z_{\xi\xi\xi}^2 + z_{\xi\xi t}^2 + z_{\xi tt}^2)\, d\xi\, dt \\
&\qquad \left. + \alpha \int_0^T (1+t)^{\alpha-1} \int_{-\infty}^{+\infty} (z_{\xi tt}^2 + z_{\xi\xi t}^2)\, d\xi\, dt \right\},
\end{aligned} \tag{55}
$$

and

$$
\begin{aligned}
&(1+T)^{\alpha} \int_{-\infty}^{+\infty} (z_{ttt}^2(\xi, T) + z_{\xi tt}^2(\xi, T))\, d\xi + \int_0^T (1+t)^{\alpha} \int_{-\infty}^{+\infty} z_{ttt}^2 \, d\xi\, dt \\
&\quad = O(1) \left\{ \int_{-\infty}^{+\infty} (z_{ttt}^2(\xi, 0) + z_{\xi tt}^2(\xi, 0))\, d\xi \right. \\
&\qquad + (\varepsilon + \delta) \int_0^T (1+t)^{\alpha} \\
&\qquad \times \int_{-\infty}^{+\infty} (z_{\xi}^2 + z_t^2 + z_{\xi\xi}^2 + z_{\xi t}^2 + z_{\xi\xi\xi}^2 + z_{\xi\xi t}^2 + z_{\xi tt}^2 + z_{ttt}^2)\, d\xi\, dt \\
&\qquad \left. + \alpha \int_0^T (1+t)^{\alpha-1} \int_{-\infty}^{+\infty} (z_{ttt}^2 + z_{\xi tt}^2)\, d\xi\, dt \right\}.
\end{aligned} \tag{56}
$$

If we now consider $(50)C^{-3}+(51)C^{-2}+(52)C^{-2}+(53)C^{-1}+(54)+(55)+(56)$, we get, choosing $C>0$ big enough,

$$
\begin{aligned}
&(1+T)^{\alpha}\int_{-\infty}^{+\infty}\sum_{j=2}^{3}|\nabla^{j}z|^{2}(\xi,T)\,d\xi+\int_{0}^{T}(1+t)^{\alpha}\int_{-\infty}^{+\infty}\sum_{j=2}^{3}|\nabla^{j}z|^{2}\,d\xi\,dt\\
&\quad=O(1)\left\{\int_{-\infty}^{+\infty}\sum_{j=1}^{3}|\nabla^{j}z|^{2}(\xi,0)\,d\xi+\int_{0}^{T}(1+t)^{\alpha}\int_{-\infty}^{+\infty}|\nabla z|^{2}\,d\xi\,dt\right.\\
&\qquad\left.+\int_{0}^{T}(1+t)^{\alpha-1}\int_{-\infty}^{+\infty}\sum_{j=1}^{3}|\nabla^{j}|^{2}\,d\xi\,dt\right\}.
\end{aligned}
$$

Together with (46), (49), this immediately implies

$$
\begin{aligned}
&(1+T)^{N}\int_{-\infty}^{+\infty}\sum_{j=0}^{3}|\nabla^{j}z|^{2}(\xi,T)\,d\xi+\int_{0}^{T}(1+t)^{N}\int_{-\infty}^{+\infty}\sum_{j=1}^{3}|\nabla^{j}z|^{2}\,d\xi\,dt\\
&\quad=O(1)\left\{\int_{-\infty}^{+\infty}\sum_{j=1}^{3}|\nabla^{j}z|^{2}(\xi,0)\,d\xi+\int_{-\infty}^{+\infty}\langle\xi\rangle^{N}\sum_{j=0}^{1}|\nabla^{j}z|^{2}(\xi,0)\,d\xi\right\}.
\end{aligned}
\tag{57}
$$

The corresponding estimates for $\nabla^{j}w$, $j=0,1,2$ can be obtained from those for $\nabla^{l}z$, $l=0,1,2,3$ using (15) and (16); in a similar way, we can express the time-derivatives of z in the right-hand side of (57) in terms of ξ-derivatives of z and w. We write this final result in terms of the original variables x, t: assuming (47), there exist positive constants ε, C, which are independent of $\bar{u}$, $\bar{v}$ given in (10), such that, for all $T>0$,

$$
\begin{aligned}
&(1+T)^{N}\int_{-\infty}^{+\infty}\left(\sum_{j=0}^{3}|\nabla^{j}z|^{2}(x,T)+\sum_{j=0}^{2}|\nabla^{j}w|^{2}(x,T)\right)dx\\
&\qquad+\int_{0}^{T}(1+t)^{N}\int_{-\infty}^{+\infty}\left(|\mu_{x}|\,z^{2}+\sum_{j=1}^{3}|\nabla^{j}z|^{2}+\sum_{j=0}^{2}|\nabla^{j}w|^{2}\right)dx\,dt\\
&\qquad+\int_{0}^{T}(1+t)^{N-1}\int_{-\infty}^{+\infty}z^{2}\,dx\,dt\\
&\quad\leqslant C\cdot\left\{\int_{-\infty}^{+\infty}(1+|x|^{N})(z^{2}+z_{x}^{2}+w^{2})(x,0)\,dx\right.\\
&\qquad\left.+\int_{-\infty}^{+\infty}\left(\sum_{j=0}^{3}\left|\frac{\partial^{j}z}{\partial x^{j}}\right|^{2}(x,0)+\sum_{j=0}^{2}\left|\frac{\partial^{j}w}{\partial x^{j}}\right|^{2}(x,0)\right)dx\right\}
\end{aligned}
$$

provided

$$\|z(\,\cdot\,,0)\|_{H^3} + \|w(\,\cdot\,,0)\|_{H^2} \leqslant \varepsilon,$$

with $\varepsilon > 0$ small.

In particular, we see that, for any $2 \leqslant p \leqslant +\infty$,

$$\sum_{j=0}^{2} \|\nabla^j z(\,\cdot\,,t)\|_{L^p} + \sum_{j=0}^{1} \|\nabla^j w(\,\cdot\,,t)\|_{L^p} = O(1)(1+t)^{-N/2}.$$

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